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RESEARCH ARTICLE

Solution of Unbounded Boundary Layer Equation using Modified Homotopy Perturbation Method

*Jamshaid ul Rahman[1], Muhammad Suleman[2], Naveed Anjum[3]
[1]Department of Basic Science, Blochistan University of Engineering and Technology, Khuzdar, Pakistan.
[2]Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan.
[3]Department of Mathematics, Government College University, Faisalabad, Pakistan.

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ABSTRACT

In this paper modified Homotopy Perturbation Method (HPM) with two expanding parameters is used to study the boundary layer flow on unbounded domain. Accordingly, after transformation the differential equations govern into singular differential equations with classical boundary conditions which can be directly solved by modified HPM. An approximate solution of unbounded boundary layer problem is obtained and results are compared with those in open literature. The results show that the modified homotopy perturbation method with two expanding parameters is more effective for nonlinear equation with two nonlinear terms than the traditional homotopy perturbation method with one expanding parameter. Solution steps of the prescribed method are simple and easy.

Keywords: Boundary layer problem, Modified HPM, Analytical solution, Unbounded domain, Traditional homotopy perturbation method.

1. INTRODUCTION

Due to fast growing field of nonlinear sciences, scientists and engineers are more interested to solve these problems with asymptotic, analytic and numerical techniques [1] for nonlinear problems in fluid mechanics, plasma physics, solid state physics and many other field of science [2]. Particularly, scientist and engineers are more interested in boundary layer problems, especially with unbounded domain. Many types of analytical methods are applied to find out the solution along with numerical solution method. Homotopy perturbation method [3] is effective for solving nonlinear problems which arises in different fields. Explicit solution of Helmotz equation and fifth order Kdv equation using HPM [4], thin film flow of third grade fluid on a moving belt was found by He's HPM [5]. A numerical solution of Blasius equation by Adomian's decomposition method and comparison with HPM [6] for bifurcation of nonlinear problems [7] and many more applications can be found out in recent literatures.

2. MODIFIED HOMOTOPY PERTURBATION METHOD WITH TWO EXPANDING PARAMETERS

Modified HPM [3] provides an alternative approach by introducing two expanding parameters along with modified Lindstedt-Poincare method and can be used to solve various nonlinear equations. The method has eliminated limitations of the traditional perturbation methods. On the other hand, it can take full advantage of the traditional perturbation techniques [8].

Some ways of constructing homotopy for nonlinear equation with multiple nonlinear terms are shown in equations (1) to (25).

$$\tilde{L}u+P_1N_1u+P_2N_2u+P_1P_2(Lu-\tilde{L}u)=0$$

$$or$$

$$\tilde{L}u+P_1N_1u+P_2N_2u+f(P_1)g(P_2)(Lu-\tilde{L}u)=0 \tag{1}$$

*Corresponding author. Tel.: +923143041122
Email address: jamshaidrahman@gmail.com (J.Rahman)
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or

$$\tilde{L}u + P_1 N_1 u + P_2 N_2 u + h(P_1, P_2)(Lu - \tilde{L}u) = 0$$

where $\tilde{L}$ is linear and $\tilde{L}u = 0$ can approximately match the solution properties. f, g and h are continuous functions satisfying the following conditions,

$$f(1)g(1) = 1, \qquad h(1,1) = 1 \tag{2}$$

This method is the natural extension of the classical homotopy perturbation method and can be used to solve the nonlinear equations with high accuracy and efficiency.

3. SOLUTION OF THE PROBLEM BY MODIFIED HPM

Consider the unbounded boundary layer problem

$$u''' + (a-1)uu'' - 2a(u')^2 = 0, \qquad a > 0 \tag{3}$$

$$u(0) = 0, \qquad u'(0) = 1, \qquad u'(\infty) = 0 \tag{4}$$

Now by using the expanding parameter method we can write the above equation as,

$$u''' + 0 \cdot u + (a-1)uu'' - 2a(u')^2 = 0 \tag{5}$$

We construct a homotopy of the above equation along with introducing a parameter expansion method, according to [1],

$$u''' + 0 \cdot u + P_1(a-1)uu'' - P_2\left(2a(u')^2\right) = 0 \tag{6}$$

where, P_1 and P_2 are homotopy parameters, $P_1 \in [0,1]$ and $P_2 \in [0,1]$. Solution and coefficients of 0 in linear term in equation (6) are expanded in the forms,

$$u = u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \dots \tag{7}$$

$$0 = \omega^3 + p_1 \omega_1 + p_2 \omega_2 + p_1^2 \omega_3 + p_1 p_2 \omega_4 + p_2^2 \omega_5 + \dots \tag{8}$$

This expansion is similar to modified Lindstedt - Poincare method with double series expansion [9].

Substituting equations (7) and (8) into equation (6), collecting the same power of $p_1^m p_2^n (m,n = 0,1,2,3\dots)$ and setting coefficients zero, we can readily obtain the following equations.

$$u_0''' + \omega^3 u_0 = 0, \qquad u_0(0) = 0, \qquad u_0'(0) = 1 \tag{9}$$

$$u_1''' + \omega^3 u_1 + \omega_1 u_0 + (a-1)u_0 u_0'' = 0 \tag{10}$$

$$u_2''' + \omega^3 u_2 + \omega_2 u_0 - 2a\left(u_0'\right)^2 = 0 \tag{11}$$

$$u_3''' + \omega^3 u_3 + \omega_3 u_0 + (a-1)\left(u_1 u_0'' + u_0 u_1''\right) = 0 \tag{12}$$

$$u_4''' + \omega^3 u_4 + \omega_4 u_0 + (a-1)\left(u_2 u_0'' + u_0 u_2''\right) - 2a\left(2u_0' u_1'\right) = 0 \tag{13}$$

$$u_5''' + \omega^3 u_5 + \omega_5 u_0 - 2a\left(2u_0' u_2'\right) = 0 \tag{14}$$

Now by solving equation (14) we have,

$$u_0''' + \omega^3 u_0 = 0, \qquad u_0(0) = 0, \qquad u_0'(0) = 1 \tag{15}$$

$$u_0(t) = \frac{1}{a}\left(1 - e^{-at}\right) \tag{16}$$

By using the value of u_0 from equation (16) we get the value of u_1,

$$u_1''' + \omega^3 u_1 + \omega_1 \frac{1}{a}\left(1 - e^{-at}\right) + (a-1)\left(\frac{1}{a}\left(1 - e^{-at}\right)\left(-ae^{-at}\right)\right) = 0,$$

$$u_1''' + \omega^3 u_1 + \frac{1}{a}\left(1 - e^{-at}\right) + \left(\omega_1 - (a-1)(-ae^{-at})\right) = 0,$$

$$\omega_1 - (a-1)\left(ae^{-at}\right) = 0,$$

$$\omega_1 = a(a-1)e^{-at} \qquad or \qquad \omega_1 = \frac{a(a-1)}{e^{at}} \tag{17}$$

So,

$$u_1(t) = \frac{a+1}{7\omega^3 (a-1)^{\frac{3}{2}}}\left(2e^{-\sqrt{a-1t}} - e^{-2\sqrt{a-1t}} - 1\right) \tag{18}$$

Now for the value of u_2,

$$u_2''' + \omega^3 u_2 + \omega_2 \frac{1}{a}\left(1 - e^{-at}\right) - 2a\left(e^{-at}\right)^2 = 0,$$

$$u_2''' + \omega^3 u_2 + \omega_2 \frac{1}{a}\left(1 - e^{-at}\right) - 2ae^{-2at} = 0,$$

$$\omega_2 = 0 \tag{19}$$

$$u_2 = \frac{1}{2a\omega^3} e^{-2at} + \frac{-t^2}{2a\omega^3} + \left(1 - \frac{1}{2a}\right)\frac{t}{\omega^3} \tag{20}$$

For u_3,

$$u_3''' + \omega^3 u_3 + \omega_3 \frac{1}{a}\left(1 - e^{-at}\right) + (a-1)\Bigg\{\left(\frac{a+1}{7\omega^3 (a-1)^{\frac{3}{2}}}\left(2e^{-\sqrt{a-1t}} - e^{-2\sqrt{a-1t}} - 1\right)\left(-ae^{-at}\right)\right) +$$

$$\left(\frac{1}{a}\left(1 - e^{-at}\right)\left(\frac{a+1}{7\omega^3 (a-1)^{\frac{3}{2}}}\left(2(a-1)e^{-\sqrt{a-1t}} - 4(a-1)e^{-2\sqrt{a-1t}} - 1\right)\right)\right)\Bigg\} = 0,$$

$$u_3''' + \omega^3 u_3 + \omega_3 \frac{1}{a}\left(1 - e^{-at}\right) - \frac{a(a-1)(a+1)}{7\omega^3 (a-1)^{\frac{3}{2}}}\left(2e^{-\sqrt{a-1t}} - e^{-2\sqrt{a-1t}} - 1\right)\left(e^{-at}\right) +$$

$$\left(\frac{1}{a}\left(1 - e^{-at}\right)\left(\frac{(a-1)(a+1)}{7\omega^3 (a-1)^{\frac{3}{2}}}\left(2(a-1)e^{-\sqrt{a-1t}} - 4(a-1)e^{-2\sqrt{a-1t}} - 1\right)\right)\right) = 0,$$

$$u_3''' + \omega^3 u_3 + \omega_3 \frac{1}{a}\left(1 - e^{-at}\right) + \left(\frac{1}{a}\left(1 - e^{-at}\right)\left(\frac{2(a+1)}{7\omega^3 \left(a-1\right)^{\frac{-1}{2}}}\left(e^{-\sqrt{a-1t}} - 2e^{-2\sqrt{a-1t}}\right)\right)\right)$$

$$-\frac{a(a+1)}{7\omega^3 (a-1)^{\frac{1}{2}}}\left(2e^{-\sqrt{a-1t}} - e^{-2\sqrt{a-1t}} - 1\right)\left(e^{-at}\right) = 0,$$

$$\omega_3 + \frac{2(a+1)}{7\omega^3 (a-1)^{\frac{-1}{2}}}\left(e^{-\sqrt{a-1t}} - 2e^{-2\sqrt{a-1t}}\right) = 0,$$

$$\omega_3 = -\left(\frac{2(a+1)}{7\omega^3\left(a-1\right)^{\frac{-1}{2}}}\left(e^{-\sqrt{a-1}t}-2e^{-2\sqrt{a-1}t}\right)\right) \tag{21}$$

So u_3 is,

$$u_3 = \frac{a(a+1)}{7\omega^6\left(a-1\right)^{\frac{1}{2}}}\left(\frac{-2e^{-\left(\sqrt{a-1}+a\right)t}}{(\sqrt{a-1}+a)^3}+\frac{e^{-\left(2\sqrt{a-1}+a\right)t}}{\left(2\sqrt{a-1}+a\right)^3}+\frac{e^{-at}}{a^3}+\frac{a(a+1)}{7\omega^3\left(a-1\right)^{\frac{1}{2}}}\right)+$$
$$\left(\frac{2a^3\left(2\sqrt{a-1}+a\right)^3-a^3\left(\sqrt{a-1}+a\right)^3-\left(\sqrt{a-1}+a\right)^3a^3\left(2\sqrt{a-1}+a\right)^3}{a^3\left(\sqrt{a-1}+a\right)^3\left(2\sqrt{a-1}+a\right)^3}\right)t+$$
$$\left(\frac{2a^2\left(2\sqrt{a-1}+a\right)^2-a^2\left(\sqrt{a-1}+a\right)^2-\left(\sqrt{a-1}+a\right)^2a^2\left(2\sqrt{a-1}+a\right)^2}{a^2\left(\sqrt{a-1}+a\right)^2\left(2\sqrt{a-1}+a\right)^2}\right)t^2 \tag{22}$$

Now by solving equation (13) and equation (14), we get the value of ω_4 and ω_5 respectively.

$$\omega_4 = \frac{a-1}{a\omega^3}\left(2a^2e^{-2at}-1\right) \tag{23}$$

$$\omega_5 = 0 \tag{24}$$

With the help of ω_4 and ω_5, we can find out the values of u_4 and u_5 respectively. By setting $p_1 = 1$ and $p_2 = 1$ in equation (7), we have the solution of the form,

$$u(t) = u_0+u_1+u_2+u_3+\ldots$$

$$= \frac{1}{a}\left(1-e^{-at}\right)+\frac{a+1}{7\omega^3\left(a-1\right)^{\frac{3}{2}}}\left(2e^{-\sqrt{a-1}t}-e^{-2\sqrt{a-1}t}-1\right)+\frac{1}{2a\omega^3}e^{-2at}+\frac{-t^2}{2a\omega^3}+\left(1-\frac{1}{2a}\right)\frac{t}{\omega^3}+$$
$$\frac{a(a+1)}{7\omega^6\left(a-1\right)^{\frac{1}{2}}}\left(\frac{-2e^{-(\sqrt{a-1}+a)t}}{(\sqrt{a-1}+a)^3}+\frac{e^{-(2\sqrt{a-1}+a)t}}{(2\sqrt{a-1}+a)^3}+\frac{e^{-at}}{a^3}\right)+$$
$$\left(\frac{2a^3\left(2\sqrt{a-1}+a\right)^3-a^3\left(\sqrt{a-1}+a\right)^3-\left(\sqrt{a-1}+a^3\right)a^3\left(2\sqrt{a-1}+a\right)^3}{a^3\left(\sqrt{a-1}+a\right)^3\left(2\sqrt{a-1}+a\right)^3}\right)t+$$
$$\left(\frac{2a^2(2\sqrt{a-1}+a)^2-a^2(\sqrt{a-1}+a)^2-(\sqrt{a-1}+a^2)a^2(2\sqrt{a-1}+a)^2}{a^2(\sqrt{a-1}+a)^2(2\sqrt{a-1}+a)^2}\right)t^2$$

$$u''(t) = -ae^{-at}+\frac{a+1}{7\omega^3\left(a-1\right)^{\frac{3}{2}}}\left(\left(-\sqrt{a-1}\right)^2e^{-\sqrt{a-1}t}-\left(-2\sqrt{a-1}\right)^2e^{-2\sqrt{a-1}t}\right)-\frac{1}{\omega^3}\left(-2ae^{-2at}-\frac{1}{a}\right)+$$
$$\frac{a\left(a+1\right)}{7\omega^6\left(a-1\right)^{\frac{1}{2}}}\left\{\left(\frac{-2e^{-\left(\sqrt{a-1}+a\right)t}}{\left(\sqrt{a-1}+a\right)}+\frac{e^{-\left(2\sqrt{a-1}+a\right)t}}{\left(2\sqrt{a-1}+a\right)}+\frac{e^{-at}}{a}\right)-\right.$$
$$\left.\left(\frac{2a^2\left(2\sqrt{a-1}+a\right)^2-a^2\left(\sqrt{a-1}+a\right)^2-\left(\sqrt{a-1}+a\right)^2a^2\left(2\sqrt{a-1}+a\right)^2}{a^2\left(\sqrt{a-1}+a\right)^2\left(2\sqrt{a-1}+a\right)^2}\right)\right\} \tag{25}$$

Table 1 given below shows the numerical value of $u''(0)$.

Table 1.Numerical value of $u''(0)$

A	Modified HPM	HPM	Pade Approximation
4	-2.1344	-2.5568	-2.483954032
10	-3.8965	-4.0476	-4.026385103
100	-12.2430	-12.8501	-12.84334315
1000	-40.1256	-40.6556	-40.65538218
5000	-89.7219	-90.9127	-104.8420672

4. CONCLUSION

In this paper we apply He's modified homotopy perturbation method with two expanding parameters to compute the approximation of $u(x)$ for different values of $n(n > 0)$. Results show that the modified HPM gives better result than traditional HPM and Pade-Approximation method, which implies that modified HPM is a reliable and an efficient method for higher order accuracy. We suggest that, this method can be further applied to nonlinear problems having multiple nonlinear terms.

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